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DUALITY OF FOURIER TYPE WITH RESPECT TO LOCALLY COMPACT ABELIAN GROUPS

BY

AICKE HINRICHS*

Department of Mathematics, Friedrich-Schiller-University Jena
Ernst-Abbe-Platz 1–4, D-07743, Germany
e-mail: nah@rz.uni-jena.de

AND

HUN HEE LEE**

Department of Mathematical Sciences, Seoul National University
San56-1 Shinrim-dong Kwanak-gu Seoul 151-747, Republic of Korea
e-mail: Lee.hunhee@gmail.com, bbking@amath.kaist.ac.kr

ABSTRACT

It is shown that a Banach space X has Fourier type p with respect to a locally compact abelian group G if and only if the dual space X' has Fourier type p with respect to G if and only if X has Fourier type p with respect to the dual group of G. This extends previously known results for the classical groups and the Cantor group to the setting of general locally compact abelian groups.

1. Introduction and notation

It is well-known that the Pontrjagin duality for locally compact abelian groups fits together with the duality of Banach spaces in the study of Hausdorff–Young inequalities for vector-valued functions. More precisely, if X is a Banach space with dual space X' and G is a locally compact abelian (in short lca) group with dual group G', then the validity of a Hausdorff–Young inequality

$$(1) \qquad \|\mathcal{F}_G f\|_{L^X_{p'}(G')} \le c\|f\|_{L^X_p(G)}$$

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for all X-valued functions f in $L_p^X(G)$ implies a Hausdorff–Young inequality

$$\|\mathcal{F}_{G'} g\|_{L_{p'}^{X'}(G)} \leq c\|g\|_{L_p^{X'}(G')} \tag{2}$$

for all $g \in L_p^{X'}(G')$. Here p is a number in the interval $[1,2]$, p' is the conjugate number given by $1/p + 1/p' = 1$, and $\mathcal{F}_G$ and $\mathcal{F}_{G'}$ denote the Fourier transforms on G and G', respectively. Proofs of this crucial fact can be found in [1, 2]. Observe that the minimal possible constants c in (1) and (2) coincide.

However, in the case of the classical groups of the real numbers $\mathbb{R}$, the Torus $\mathbb{T}$ and the integers $\mathbb{Z}$, and in the case of the Cantor group $\mathbb{D} = \{0,1\}^{\mathbb{N}}$, even more is true. These groups exhibit an autoduality with respect to vector-valued Hausdorff–Young inequalities in the following sense. If (1) is satisfied for all X-valued functions f in $L_p^X(G)$, where G is one of these groups, then it is also satisfied for all X'-valued functions f in $L_p^{X'}(G)$ with possibly different constant c. This is obvious for $G = \mathbb{R}$ for $\mathbb{R}' = \mathbb{R}$ and is proved for $G = \mathbb{T}, \mathbb{Z}$ in [1, 2, 4] and for $G = \mathbb{D}$ in [2].

The main purpose of this paper is to show that this autoduality behavior is not special to the above-mentioned lca groups but is actually true for any lca group G. The question of a characterization of lca groups with this autoduality was also raised in [2]. Before we state the result more formally, let us introduce the necessary notation.

We will work in the framework of an lca group G which comes equipped with its Haar measure μ_G. X and Y will always be Banach spaces and T a linear and bounded operator from X to Y. The dual operator of T is T'. For $p \in [1,\infty]$, $L_p(G)$ stands for the Lebesgue space $L_p(G,\mu)$ and $L_p^X(G)$ is the Bochner–Lebesgue space of p-integrable X-valued functions. For a function $f \in L_1(G)$, the Fourier transform $\mathcal{F}_G f$ is given by

$$\mathcal{F}_G(f)(\gamma) = \int_G f(x)\overline{\gamma(x)}d\mu_G(x)$$

for $\gamma \in G'$. It is a function in $C_0(G')$, the space of continuous functions on G' vanishing at infinity. The classical Hausdorff–Young inequality for G makes it possible to define the Fourier transform also for functions in $L_p(G)$ for $1 < p \leq 2$ so that $\mathcal{F}_G$ defines a bounded operator from $L_p(G)$ into $L_{p'}(G')$. We assume the standard normalization of the Haar measure μ_G so that Plancherel's Theorem holds which means that $\mathcal{F}_G$ is an isometry from $L_2(G)$ onto $L_2(G')$.

From now on, p will always be in the interval $[1,2]$. The Banach space X is said to have Fourier type p with respect to G if the Fourier transform $\mathcal{F}_G$, which is originally defined in the obvious way on the algebraic tensor product $L_p(G) \otimes X$ of finite sums $\sum \varphi_k x_k$ with $\varphi_k \in L_p(G)$ and $x_k \in X$, extends to a bounded linear operator from $L_p^X(G)$ into $L_{p'}^X(G')$. This just means that (1) holds. The operator norm of this operator is denoted by $\|X|\mathcal{FT}_p^G\|$. This concept was first introduced by Peetre in [7] for $G = \mathbb{R}$ and by Milman in [5] in the general case.

This definition is easily extended to the case of operators. We will say that T has Fourier type p with respect to G if the operator $\mathcal{F}_G \otimes T$ originally defined from $L_p(G) \otimes X$ to $L_{p'}(G') \otimes Y$ extends to a bounded linear operator from $L_p^X(G)$ to $L_{p'}^Y(G')$. Again, the operator norm is then denoted by $\|T|\mathcal{FT}_p^G\|$, so that $\|I_X|\mathcal{FT}_p^G\| = \|X|\mathcal{FT}_p^G\|$ where I_X is the identity operator on X. The class of all operators of Fourier type p with respect to G equipped with the norm $\|\cdot|\mathcal{FT}_p^G\|$ forms a Banach operator ideal in the sense of Pietsch; see [8]. We denote this ideal by $\mathcal{FT}_p^G$. Observe that every operator has Fourier type 1 with respect to any lca group, but that for any infinite lca group G and any $p \in (1, 2]$, there are Banach spaces without Fourier type p with respect to G. For examples and more information on the notion of Fourier type, we refer the reader to [1, 2, 8].

The standard duality result mentioned in the introductory paragraph can now be formulated as follows. The generalization of the proofs in [1, 2] to the operator case is straightforward.

Theorem 1: *A bounded linear operator T has Fourier type p with respect to the lca group G if and only if the dual operator T' has Fourier type p with respect to the dual group G'. Moreover, in this case $\|T|\mathcal{FT}_p^G\| = \|T'|\mathcal{FT}_p^{G'}\|$.*

In the language of operator ideals, this says that the dual operator ideal of $\mathcal{FT}_p^G$ is $\mathcal{FT}_p^{G'}$ with equal norms.

Now we can formulate the main result of this paper, which says that the operator ideal $\mathcal{FT}_p^G$ is even symmetric.

Theorem 2: *For any bounded linear operator T between Banach spaces and all lca groups G, the following properties are equivalent:*

(i) *T has Fourier type p with respect to G.*
(ii) *T' has Fourier type p with respect to G.*
(iii) *T has Fourier type p with respect to G'.*
(iv) *T' has Fourier type p with respect to G'.*

In the next section, we collect some results on Fourier type norms for product groups which are needed for the proof of Theorem 2. Section 3 contains the proof, which is first carried out for the case that G is compact or discrete. Then the result can be extended to general lca groups G with the help of the structure theory of lca groups.

2. Fourier type with respect to product groups

We start by providing a theorem connecting the Fourier type of an operator T with respect to a product group $G \times H$ with the Fourier type of the tensor product $\mathcal{F}_H \otimes T\colon L_p^X(H) \to L_{p'}^Y(H')$. It was observed in [1] that $T \in \mathcal{FT}_p^{G\times H}$ already implies $T \in \mathcal{FT}_p^H$, so that the latter operator is well defined. Again, in [1] only spaces are considered, but the extension of the proof to the operator case is straightforward.

THEOREM 3: *Let G and H be lca groups and $1 < p \leq 2$. Then $T \in \mathcal{FT}_p^{G\times H}$ if and only if $T \in \mathcal{FT}_p^H$ and $\mathcal{F}_H \otimes T\colon L_p^X(H) \to L_{p'}^Y(H') \in \mathcal{FT}_p^G$. Moreover, in this case*

$$\|\mathcal{F}_H \otimes T | \mathcal{FT}_p^G\| = \|T|\mathcal{FT}_p^{G\times H}\|.$$

Proof: Assume first that $T \in \mathcal{FT}_p^H$ and $\mathcal{F}_H \otimes T\colon L_p^X(H) \to L_{p'}^Y(H') \in \mathcal{FT}_p^G$. We will prove that

$$\|T|\mathcal{FT}_p^{G\times H}\| \leq \|\mathcal{F}_H \otimes T|\mathcal{FT}_p^G\| =: c. \tag{3}$$

This implies that $T \in \mathcal{FT}_p^{G\times H}$. By density, to verify (3) it is enough to show that

$$\Big\| \sum_j (\mathcal{F}_{G\times H}\varrho_j)Tx_j \Big\|_{L_{p'}^Y(G'\times H')} \leq c \Big\| \sum_j \varrho_j x_j \Big\|_{L_p^X(G\times H)}$$

holds for all finite families $\varrho_j \in L_p(G\times H)$ and $x_j \in X$. Again by density of $L_p(G)\otimes L_p(H)$ in $L_p(G\times H)$ this is equivalent to the inequality

$$\Big\| \sum_j \mathcal{F}_{G\times H}(\varphi_j\psi_j)Tx_j \Big\|_{L_{p'}^Y(G'\times H')} \leq c \Big\| \sum_j \varphi_j\psi_j x_j \Big\|_{L_p^X(G\times H)}$$

for all $\varphi_j \in L_p(G)$, $\psi_j \in L_p(H)$ and $x_j \in X$. Since

$$\mathcal{F}_{G\times H}(\varphi_j\psi_j)Tx_j = (\mathcal{F}_G\varphi_j)(\mathcal{F}_H\psi_j)Tx_j = (\mathcal{F}_G\varphi_j)(\mathcal{F}_H\otimes T)(\psi_j\otimes x_j) \tag{4}$$

and

$$\Big\| \sum_j \varphi_j\psi_j x_j \Big\|_{L_p^X(G\times H)} = \Big\| \sum_j \varphi_j(\psi_j\otimes x_j) \Big\|_{L_p^{L_p^X(H)}(G)}, \tag{5}$$

this is immediate from the definition of $c = \|\mathcal{F}_H \otimes T|\mathcal{FT}_p^G\|$.

Now assume that $T \in \mathcal{FT}_p^{G\times H}$. By the remark before the statement of the theorem, we know that $T \in \mathcal{FT}_p^{H}$. We will prove that

$$\|\mathcal{F}_H \otimes T|\mathcal{FT}_p^{G}\| \leq \|T|\mathcal{FT}_p^{G\times H}\| =: d. \tag{6}$$

Again density arguments reduce this to the question whether

$$\Big\|\sum_j (\mathcal{F}_G\varphi_j)(\mathcal{F}_H\otimes T)(\psi_j\otimes x_j)\Big\|_{L_{p'}^{L_{p'}^{Y}(H')}(G')} \leq d\Big\|\sum_j \varphi_j\psi_j x_j\Big\|_{L_p^{L_p^{X}(H)}(G)}$$

holds for all finite families $\varphi_j \in L_p(G)$, $\psi_j \in L_p(H)$ and $x_j \in X$. The proof of (6) is finished by the observation that the definition of $d = \|T|\mathcal{FT}_p^{G\times H}\|$ together with (4), (5) and

$$\Big\|\sum_j (\mathcal{F}_G\varphi_j)(\mathcal{F}_H\otimes T)(\psi_j\otimes x_j)\Big\|_{L_{p'}^{L_{p'}^{Y}(H')}(G')} = \Big\|\sum_j (\mathcal{F}_G\varphi_j)(\mathcal{F}_H\psi_j)Tx_j\Big\|_{L_{p'}^{Y}(G'\times H')}$$

imply this inequality. Finally, (3) and (6) also show the claimed equality of norms. ∎

Corollary 4: *Let G_1, G_2 and H be lca groups and $1 < p \leq 2$. If there exists a constant c such that*

$$\|T|\mathcal{FT}_p^{G_2}\| \leq c\|T|\mathcal{FT}_p^{G_1}\|$$

for all operators $T \in \mathcal{FT}_p^{G_1}$ then also

$$\|T|\mathcal{FT}_p^{G_2\times H}\| \leq c\|T|\mathcal{FT}_p^{G_1\times H}\|$$

for all operators $T \in \mathcal{FT}_p^{G_1\times H}$.

Proof: If $T \in \mathcal{FT}_p^{G_1\times H}$, we obtain from Theorem 3 that $\mathcal{F}_H \otimes T \in \mathcal{FT}_p^{G_1}$ and

$$\|T|\mathcal{FT}_p^{G_1\times H}\| = \|\mathcal{F}_H \otimes T|\mathcal{FT}_p^{G_1}\|.$$

Now the assumption implies that $\mathcal{F}_H \otimes T \in \mathcal{FT}_p^{G_2}$ and

$$\|\mathcal{F}_H \otimes T|\mathcal{FT}_p^{G_2}\| \leq c\|\mathcal{F}_H \otimes T|\mathcal{FT}_p^{G_1}\|.$$

Applying Theorem 3 once more, we find that $T \in \mathcal{FT}_p^{G_2\times H}$ and

$$\|T|\mathcal{FT}_p^{G_2\times H}\| = \|\mathcal{F}^H \otimes T|\mathcal{FT}_p^{G_2}\|.$$

Altogether, we proved the claim. ∎

From this corollary, we immediately obtain generalizations of the results of [1]. These generalizations were already observed in [6], where the results are shown using close analogues of the proofs in [1, 2].

Corollary 5: *For any $d = 1, 2, \ldots, \infty$ and all lca groups G, the ideals $\mathcal{FT}_p^{\mathbb{Z}^d \times G}$ and $\mathcal{FT}_p^{\mathbb{Z} \times G}$ coincide with equal norms. Also, the ideals $\mathcal{FT}_p^{\mathbb{T}^d \times G}$ and $\mathcal{FT}_p^{\mathbb{T} \times G}$ coincide with equal norms.*

Proof: Use Corollary 4 together with Theorem 1.7 in [1] and the following corollaries there. ∎

For the next corollaries, we need the Babenko–Beckner constants given by $B_p = \sqrt{p^{1/p}/p'^{1/p'}}$ for $1 < p \leq 2$.

Corollary 6: *For any $d = 1, 2, \ldots$ and all lca groups G, the ideals $\mathcal{FT}_p^{\mathbb{R}^d \times G}$ and $\mathcal{FT}_p^{\mathbb{Z}^d \times G}$ coincide and the corresponding norms satisfy the inequalities*

$$\|T|\mathcal{FT}_p^{\mathbb{R}^d \times G}\| \leq \|T|\mathcal{FT}_p^{\mathbb{Z}^d \times G}\| \leq B_p^{-d}\|T|\mathcal{FT}_p^{\mathbb{R}^d \times G}\|.$$

Proof: Use Corollary 4 together with the corresponding corollary in [1]. ∎

Corollary 7: *For any $d = 1, 2, \ldots$ and all lca groups G, the ideals $\mathcal{FT}_p^{\mathbb{Z}^d \times G}$ and $\mathcal{FT}_p^{\mathbb{T}^d \times G}$ coincide and the corresponding norms satisfy the inequalities*

$$B_p\|T|\mathcal{FT}_p^{\mathbb{T}^d \times G}\| \leq \|T|\mathcal{FT}_p^{\mathbb{Z}^d \times G}\| \leq B_p^{-1}\|T|\mathcal{FT}_p^{\mathbb{Z}^d \times G}\|.$$

Proof: Use Corollary 4 together with Theorem 1.8 in [1]. ∎

3. Proof of the duality theorem

Since Theorem 2 is trivial for $p = 1$, we always assume $1 < p \leq 2$ in this section. We start by recalling the following result from Lemma 3.2 and Proposition 3.3 from [1]. Here $C_c(H, X)$ denotes the space of all compactly supported X-valued functions on the lca group H.

Proposition 8: *Let G be an lca group which contains an open subgroup H. Then $\mathcal{FT}_p^G \subset \mathcal{FT}_p^H$ and*

$$\|T|\mathcal{FT}_p^H\| \leq \|T|\mathcal{FT}_p^G\| \quad \text{for all } T \in \mathcal{FT}_p^G.$$

Furthermore, for any $f \in C_c(H, X)$, the extension g to all of G defined by zero outside of H satisfies

$$\frac{\|\mathcal{F}_G \otimes Tg\|_{L_{p'}^Y(G')}}{\|g\|_{L_p^X(G)}} = \frac{\|\mathcal{F}_H \otimes Tf\|_{L_{p'}^Y(H')}}{\|f\|_{L_p^X(H)}}.$$

Now we can prove our main duality result in the case that G is compact or discrete.

Theorem 9: *Let G be a compact or discrete abelian group. Then $\mathcal{FT}_p^{G'} \subset \mathcal{FT}_p^G$ and*

$$\|T|\mathcal{FT}_p^G\| \le B_p^{-1}\|T|\mathcal{FT}_p^{G'}\|$$

for all $T \in \mathcal{FT}_p^{G'}$.

Proof: By the standard duality Theorem 1 it is sufficient to consider the case that G is discrete. By a density argument, it is then enough to prove

$$\|(\mathcal{F}_G \otimes T)g\|_{L_{p'}^Y(G')} \le B_p^{-1}\|T|\mathcal{FT}_p^{G'}\|\|g\|_{L_p^X(G)} \tag{7}$$

for functions g with finite support on G. So suppose that g is such a function with finite support $S \subset G$ given by $g(a) = x_a \in X$ for $a \in S$ and $g(a) = 0$ for $a \in G \setminus S$. The finitely generated group $H = \langle S \rangle$ is topologically isomorphic to $\mathbb{Z}^d \times F$ for some $d \in \mathbb{N} \cup \{0\}$ and some finite abelian group F. Then we can consider the restriction f of g to H. By Proposition 8, we obtain that (7) is equivalent to

$$\|(\mathcal{F}_H \otimes T)f\|_{L_{p'}^Y(H')} \le B_p^{-1}\|T|\mathcal{FT}_p^{G'}\|\|f\|_{L_p^X(H)}. \tag{8}$$

To prove this inequality, we first observe that by the definition of Fourier type with respect to H

$$\|(\mathcal{F}_H \otimes T)f\|_{L_{p'}^Y(H')} \le \|T|\mathcal{FT}_p^H\|\|f\|_{L_p^X(H)}.$$

Corollary 5 and Corollary 7 imply that

$$\|T|\mathcal{FT}_p^H\| = \|T|\mathcal{FT}_p^{\mathbb{Z}^d\times F}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}\times F}\| \le B_p^{-1}\|T|\mathcal{FT}_p^{\mathbb{T}\times F}\|.$$

Since F is finite the dual group of F is isomorphic to F. Then the standard duality Theorem 1 and Corollary 5 give

$$\|T|\mathcal{FT}_p^{\mathbb{T}\times F}\| = \|T'|\mathcal{FT}_p^{\mathbb{Z}\times F}\| = \|T'|\mathcal{FT}_p^{\mathbb{Z}^d\times F}\| = \|T'|\mathcal{FT}_p^H\|.$$

Since G is discrete, H is an open subgroup of G. Applying Proposition 8 then implies together with the standard duality Theorem 1

$$\|T'|\mathcal{FT}_p^H\| \le \|T'|\mathcal{FT}_p^G\| = \|T|\mathcal{FT}_p^{G'}\|.$$

Altogether, we obtain (8) which completes the proof of the theorem. ∎

The preceding theorem is already sufficient to prove Theorem 2 for the special case of compact or discrete abelian groups. The next theorem brings us a step closer to the general case, which will then follow from the structure theorem of lca groups.

Theorem 10: *Let G be an lca group which has a compact and open subgroup. Then $\mathcal{FT}_p^{G'} \subset \mathcal{FT}_p^{G}$ and*

$$\|T|\mathcal{FT}_p^{G}\| \le B_p^{-3}\|T|\mathcal{FT}_p^{G'}\| \quad \text{for all } T \in \mathcal{FT}_p^{G'}. \tag{9}$$

Proof: Let us first assume that G is topologically isomorphic to $\mathbb{Z}^d \times K$ for some $d \in \mathbb{N} \cup \{0\}$ and some compact abelian group K. Using in succession Corollaries 5, 7, Theorem 9, and Corollaries 7, 5 again, we get

$$\begin{aligned}\|T|\mathcal{FT}_p^{G}\| &= \|T|\mathcal{FT}_p^{\mathbb{Z}^d\times K}\| = \|T|\mathcal{FT}_p^{\mathbb{Z}\times K}\| \le B_p^{-1}\|T|\mathcal{FT}_p^{\mathbb{T}\times K}\| \\ &\le B_p^{-2}\|T|\mathcal{FT}_p^{\mathbb{Z}\times K'}\| \le B_p^{-3}\|T|\mathcal{FT}_p^{\mathbb{T}\times K'}\| = B_p^{-3}\|T|\mathcal{FT}_p^{\mathbb{T}^d\times K'}\| \\ &= B_p^{-3}\|T|\mathcal{FT}_p^{G'}\|.\end{aligned}$$

Now assume the general case that we just have an open and compact subgroup H of G. Let $q: G \to G/H$ be the canonical quotient map and observe that G/H is a discrete abelian group. By a density argument, it is enough to show that

$$\|(\mathcal{F}_G \otimes T)g\|_{L^Y_{p'}(G')} \le B_p^{-3}\|T|\mathcal{FT}_p^{G'}\|\|g\|_{L^X_p(G)} \tag{10}$$

for functions g vanishing outside of $q^{-1}(S)$ for some finite set $S \subset G/H$. So let g be such a function and let $M = q^{-1}(\langle S\rangle)$ be the preimage of the finitely generated group $\langle S\rangle$ which is topologically isomorphic to $\mathbb{Z}^d \times F$ for some $d \in \mathbb{N} \cup \{0\}$ and some finite abelian group F. Then M is an open subgroup of G containing H as a subgroup and M/H is topologically isomorphic to $\mathbb{Z}^d \times F$. Now it follows from the proof of Theorem 9.8 in [3] that M itself is topologically isomorphic to $\mathbb{Z}^d \times K$ for some compact abelian group K, so we already proved (9) for M.

Let f be the restriction of g to M. We apply (9) for M to obtain that

$$\|(\mathcal{F}_M \otimes T)f\|_{L^Y_{p'}(M')} \le \|T|\mathcal{FT}_p^{M}\|\|f\|_{L^X_p(M)} \le B_p^{-3}\|T|\mathcal{FT}_p^{M'}\|\|f\|_{L^X_p(M)}.$$

Using the standard duality Theorem 1 and Proposition 8 we find that

$$\|T|\mathcal{FT}_p^{M'}\| = \|T'|\mathcal{FT}_p^{M}\| \le \|T'|\mathcal{FT}_p^{G}\| = \|T|\mathcal{FT}_p^{G'}\|.$$

Hence

$$\|(\mathcal{F}_M \otimes T)f\|_{L^Y_{p'}(M')} \le \|T|\mathcal{FT}_p^{G'}\|\|f\|_{L^X_p(M)}.$$

Since g vanishes outside of M, (10) follows by Proposition 8, which concludes the proof. ∎

Now we are prepared for the proof of our main theorem.

Proof of Theorem 2: The equivalence of (i) and (iv) as well as the equivalence of (ii) and (iii) follows from the standard duality Theorem 1. Then the theorem is proved if we show that (iii) implies (i) for again this also gives that (ii) implies (iv) by Theorem 1.

So it is enough to show for any lca group G and all operators $T \in \mathcal{FT}_p^{G'}$ that

$$\|T|\mathcal{FT}_p^G\| \leq C\|T|\mathcal{FT}_p^{G'}\| \tag{11}$$

for some constant C depending only on p and G. We already know this from Theorem 10 if G contains a compact and open subgroup. The general case will now follow from the structure theorem for lca groups saying that a general lca group is topologically isomorphic to $\mathbb{R}^d \times H$ for some $d \in \mathbb{N} \cup \{0\}$ and some lca group H which has a compact and open subgroup. We obtain from Corollary 6 and Theorem 1 that

$$\|T|\mathcal{FT}_p^G\| = \|T|\mathcal{FT}_p^{\mathbb{R}^d \times H}\| \leq \|T|\mathcal{FT}_p^{\mathbb{T}^d \times H}\|.$$

Since H contains a compact and open subgroup, the same is true for $\mathbb{T}^d \times H$. Applying Theorem 10, we find that

$$\|T|\mathcal{FT}_p^{\mathbb{T}^d \times H}\| \leq B_p^{-3}\|T|\mathcal{FT}_p^{\mathbb{Z}^d \times H'}\|.$$

Finally, Corollary 6 gives that

$$\|T|\mathcal{FT}_p^{\mathbb{Z}^d \times H'}\| \leq B_p^{-d}\|T|\mathcal{FT}_p^{\mathbb{R}^d \times H'}\| = B_p^{-d}\|T|\mathcal{FT}_p^{G'}\|,$$

so that (11) indeed holds with $C = B_p^{-d-3}$. ∎

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